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Modal Truncation for Flexible Spacecraft

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A hierarchy of dynamical models is identified for large nonspinning flexible spacecraft. At each level, techniques are explained for reducing the order of the model before proceeding to the next level. These techniques have in common the presupposition that the model has at each stage been expressed in terms of its natural modes, some of which can, if necessary, be deleted based on the evaluation of one or more of the quantitative criteria proposed. These criteria are based on insights from several different perspectives, including inertial completeness, frequency relationships, controllability and observability considerations, and the contributions of individual modes to a mission-dependent cost functional (modal cost analysis). With the aid of these criteria, many of the engineering judgments related to model order reduction can be made on a rigorous quantitative basis.

Introduction

DEVELOPMENT of mathematical models suitable for the design of control systems for modern spacecraft requires the application of (at least) the following three branches of applied science: dynamics, structural dynamics, and control theory. Although these disciplines may be said to have reached individually a state of relative maturity, their unification into coherent design methodologies for spacecraft control systems continues to undergo rapid development.

Within the preview of dynamics, several formulations are available[1-4] for constructing motion equations for interconnected systems of rigid bodies. In some models, the number of degrees of freedom can be quite large. The scope of these formulations is further broadened when the constituent "bodies" in the model are assumed to experience structural deformations.[5-8] The efficient accountancy procedures of matrix structural analysis (particularly well adapted to truss-like structures), the capabilities of finite-element methods (for complicated continua), and the classical techniques of partial differential equations (for simple continua) are among the more important methods for modeling flexible space structures or substructures. In each case, the number of configuration variables (degrees of freedom) added to the model can be quite large. To control such a system leads naturally to multivariable control theory which provides potent techniques for composing effective control policies based on several inputs (measurements) and several outputs (control actions). Most multivariable control methodologies can be applied, in principle, to dynamical models of any size. In practice, however, the implied computational requirements become economically unjustifiable or numerically intractable for models above a certain order. This is true both for "off-line" (ground-based, not-real-time) digital computations, analog simulation, and for "on-line" (processing on board the spacecraft in real time) computations, although the maximum allowable model size naturally depends on circumstances. This paper is based on the premise that procedures are needed for reducing the "large" order of the dynamical models used for "large" flexible spacecraft. Several new criteria are introduced and these are combined with more familiar ones in an ordered sequence.

Generic Model

The generic model on which the specifics in this paper are based is shown in Fig. 1. The vehicle $\mathcal{V}$ comprises a rigid body $\mathcal{R}$ and N almost-elastic bodies $\mathcal{E}_1, \ldots, \mathcal{E}_N$. The latter, which are assumed to be cantilevered to $\mathcal{R}$, are said to be "almost" elastic because although their static deformations obey a linear stress-strain law, a degree of energy dissipation is imputed to them when their deformations are time dependent. The origins of this dissipation include material damping and friction due to relative motion at joints.

The generic model of Fig. 1 can be either specialized or generalized with no basic change in the model-reduction schemes proposed in this paper. If, for example, a particular spacecraft is most naturally modeled as a single flexible structure, one simply sets $N=1$ and shrinks the size of $\mathcal{R}$ to zero. On the other hand, it can be stated (although no proof is offered in this paper) that the techniques discussed subsequently remain valid—with appropriate symbolic reinterpretation but no conceptual change—if the spacecraft possesses a general topology of arbitrarily interconnected rigid and flexible bodies.

The objectives of this paper concerning the modeling of the spacecraft shown in Fig. 1 can be described in terms of the

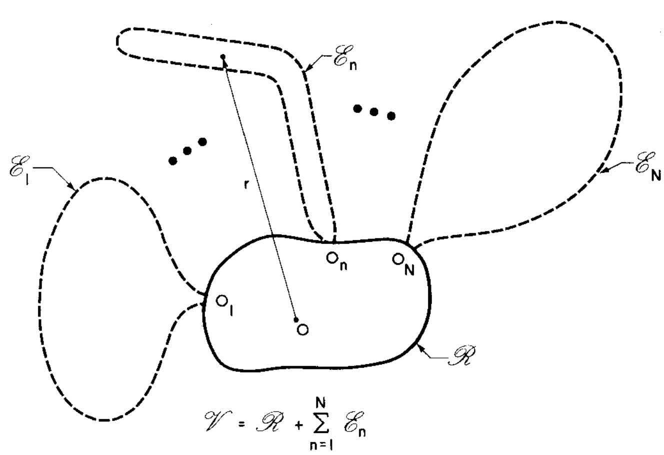

Fig. 1 Generic spacecraft model.

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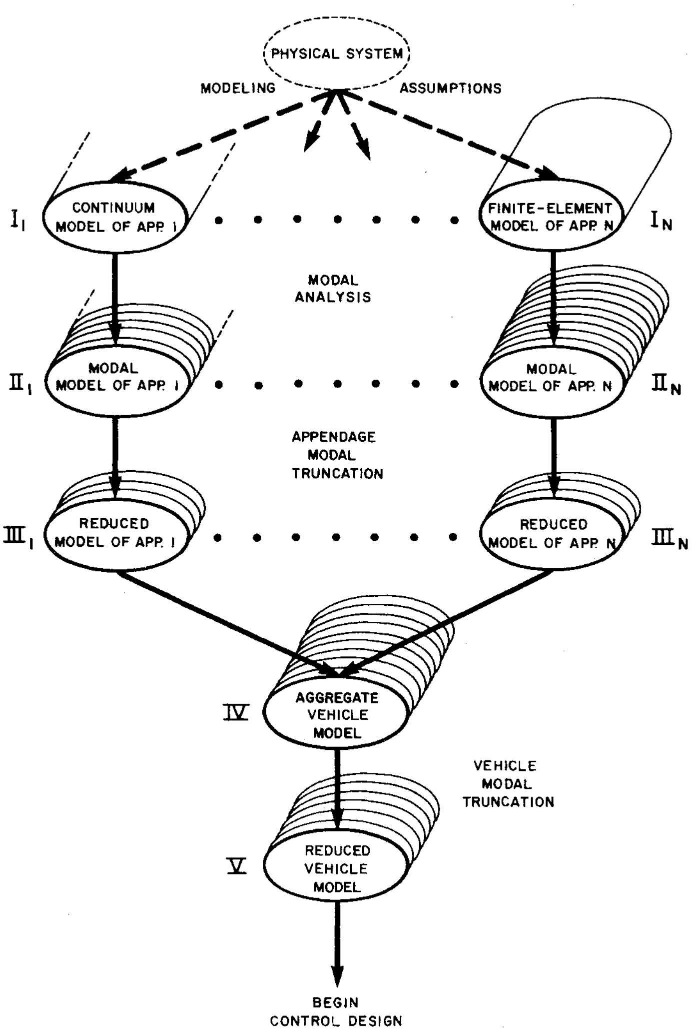

Fig. 2 Flowchart showing sequence of dynamical models (note that model order reduction can occur in several stages).

procedure outlined in Fig. 2. After the customary modeling assumptions, including small deflections and linear elasticity, one arrives at a dynamical model for each appendage (model I). Two examples of such a model are mentioned in Fig. 2: "continuum model," represented by a system of partial differential equations, and a finite-element model. The latter has a finite number of degrees of freedom (although usually a very large number), while the continuum model has, in principle, an infinite number of degrees of freedom. Looking ahead toward possible model reduction, one can then perform a modal analysis for each appendage, forming model II. At this stage, the possibility of modal truncation arises. For example, if one has ten appendages, each with at least 500 modes, one may wish to represent each appendage by not more than 100 modes (1000 modes overall). These reduced models are labeled III in Fig. 2. This paper addresses techniques for accomplishing such a truncation. A similar (further) truncation may be necessary for the aggregate spacecraft model (i.e., a reduction from the 1000 modes of model IV) and this truncation, to form the reduced model V, is also examined in this paper. Of course, if the spacecraft is best modeled as a single flexible body (or for other reasons), one may wish to proceed directly from the "physical system" to the "aggregate model," thus dispensing with models I, II, and III and the intermediate modal analyses and truncations for individual substructures of the spacecraft. The principal technical contributions of this paper are to propose new modal truncation criteria, both at the appendage stage and at the aggregate stage, and to organize these and several techniques already available into a coherent methodology. In some cases more than one alternative basis for truncation is given. For the criteria mentioned in the latter part this paper (and furthest down the flowchart in Fig. 2), information about the control system is assumed known. It is shown how to incorporate data on controller bandwidth, sensor and actuator locations, and control objectives into model reduction decisions. The problem of further model reduction for the closed-loop system, however, forms a sequel[9] to the flowchart of Fig. 2, and is beyond the scope of this paper. Closed-loop methods require substantial computation and are, therefore, not normally feasible for models as large as treated in this paper. The overall strategy, then, is to use the methods of this paper for early truncations of very large models and to use closed-loop methods in the final stages of design.

Under assumptions more fully described elsewhere,[10] the motion equations for $\mathcal{V}$ are

$$m\ddot{w}_c + \sum_{n=1}^{N} P_{n\delta}^T \ddot{\delta}_n = F(t) \tag{1}$$

$$I\ddot{\theta} + \sum_{n=1}^{N} H_{n\delta}^T \ddot{\delta}_n = G(t) \tag{2}$$

$$P_{n\delta}\ddot{w}_c + H_{n\delta}\ddot{\theta} + M_n\ddot{\delta}_n + D_n\dot{\delta}_n + K_n\delta_n = \Upsilon_{n\delta}(t) \qquad (n=1,\ldots,N) \tag{3}$$

where w_c and θ are the (small) translational and rotational displacements of $\mathcal{R}$, m and I are the mass and centroidal inertia matrix of $\mathcal{V}$, and δ_n is a column matrix whose elements are the deformational coordinates (discrete, distributed, or hybrid) for $\mathcal{E}_n$. Equation (1) is the translational motion equation for $\mathcal{V}$ and is driven by forces $F(t)$ (controls and disturbances). Equation (2) is the rotational motion equation for $\mathcal{V}$ with excitation being provided by the torque $G(t)$. The remaining Eqs. (3) specify how the δ_n, $n=1,\ldots,N$, are excited by motions of $\mathcal{R}$ (w_c and θ) and the forces and torques on $\mathcal{E}_n$, the latter represented by the modal excitation $\Upsilon_{n\delta}$, ($n=1,\ldots,N$). The mass and stiffness matrices M_n and K_n are established by appropriate structural analysis and are such that $M_n^T = M_n > 0$, $K_n^T = K_n > 0$ (positive definiteness). Equally valid procedures have not yet been developed for calculating the damping matrix D_n. The constant column matrices $P_{n\delta}$ and $H_{n\delta}$ ($n=1,\ldots,N$) couple the rigid degrees of freedom (w_c and θ) to the deformational degrees of freedom ($\delta_1,\ldots,\delta_N$).

Appendage Modal Coordinate Truncation

Possibilities for reducing the order of the dynamical system [Eqs. (1-3)] are now examined. Suppose N_n deformational degrees of freedom for appendage $\mathcal{E}_n$ are present. The total number of configuration variables is, therefore,

$$6 + \sum_{n=1}^{N} N_n \tag{4}$$

A reduction $N_n \to \bar{N}_n < N_n$ would be welcome. With Eqs. (1-3) in their present form, it is not clear how to bring this reduction about. One cannot simply delete a coordinate in δ_n and omit its associated motion equation.

A common technique, of course, is modal analysis, which is both practical and mathematically rigorous. In this way, the characteristic motions of $\mathcal{E}_n$ are identified, some of which may be arguably unessential to the intended purpose of the model. The properties of M_n and K_n ensure that there exists a transformation, denoted T_n, such that

$$T_n^T M_n T_n = 1 \qquad T_n^T K_n T_n = \Omega_n^2 \tag{5}$$

where 1 is the $N_n \times N_n$ unit matrix and $\Omega_n = \mathrm{diag}\{\Omega_{n1},\ldots,\Omega_{nN_n}\}$ contains the natural undamped vibration frequencies of $\mathcal{E}_n$. Setting, in Eqs. (1-3),

$$\delta_n(t) = T_n Q_n(t) \tag{6}$$

where Q_n is a column matrix of appendage modal coordinates, produces [after premultiplication of Eq. (3) by T_n^T]

$$m\ddot{w}_c + \sum_{n=1}^{N} P_n^T \ddot{Q}_n = F(t) \tag{7}$$

$$I\ddot{\theta} + \sum_{n=1}^{N} H_n^T \ddot{Q}_n = G(t) \tag{8}$$

$$P_n \ddot{w}_c + H_n \ddot{\theta} + \ddot{Q}_n + \hat{D}_n \dot{Q}_n + \Omega_n^2 Q_n = \Upsilon_n(t) \qquad (n=1,...,N) \tag{9}$$

where

$$P_n \triangleq T_n^T P_{n\delta} \qquad H_n \triangleq T_n^T H_{n\delta} \qquad \Upsilon_n \triangleq T_n^T \Upsilon_{n\delta} \qquad \hat{D}_n \triangleq T_n^T D_n T_n \tag{10}$$

With the reasonable assumption that $D_n^T = D_n > 0$, it is seen from the latter equation of Eqs. (10) that $\hat{D}_n$ possesses these same properties. Another property $\hat{D}_n$ inherits from D_n is its uncertainty.

With the model written in the form Eqs. (7-9), several strategies for reducing model order are suggested in the following. This order reduction is made possible by the deletion of certain elements in Q_n and their associated equations. Such methods are called "order reduction by appendage modal coordinate truncation."

Appendage Modal Completeness Index

It is known[11] that the modal parameters (P_n, H_n) ideally satisfy the conditions

$$P_n^T P_n = m_n 1 \tag{11}$$

$$H_n^T P_n = \tilde{c}_n \tag{12}$$

$$H_n^T H_n = J_n \tag{13}$$

where m_n is the mass of $\mathcal{E}_n$, 1 is the 3×3 unit matrix, c_n is the first moment of inertia of $\mathcal{E}_n$ about 0 (see Fig. 1) given by $m_n x$ (vector from 0 to mass center of $\mathcal{E}_n$), $\tilde{c}_n$ is the cross-product operator on c_n, and J_n is the (second) moment of inertia of $\mathcal{E}_n$ about 0. Conditions Eqs. (11-13) are achieved only for a *complete* set of modes. For an incomplete set (the normal case), the symmetric matrices

$$M_{rn} - \begin{bmatrix} P_n^T P_n & P_n^T H_n \\ H_n^T P_n & H_n^T H_n \end{bmatrix} \qquad (n=1,...,N_n) \tag{14}$$

are positive definite, where

$$M_{rn} \triangleq \begin{bmatrix} m_n 1 & -\tilde{c}_n \\ \tilde{c}_n & J_n \end{bmatrix} \tag{15}$$

is the 6×6 rigid mass matrix for $\mathcal{E}_n$. For symmetric positive semidefinite matrices, the spectral norm, denoted by the operator ρ, is the largest of the six real non-negative eigenvalues. It is convenient in assessing completeness to normalize the spectral norm so that $\rho[\cdot] = 1$ means "totally complete." This normalization is accomplished by premultiplying and postmultiplying the matrices in Eq. (14) by $M_{rn}^{-1/2}$, and defining the *inertial completeness index* for $\mathcal{E}_n$, as follows:

$$\mathcal{J}_n \triangleq \rho \left[M_{rn}^{-1/2} \begin{bmatrix} P_n^T P_n & P_n^T H_n \\ H_n^T P_n & H_n^T H_n \end{bmatrix} M_{rn}^{-1/2} \right] \tag{16}$$

There is no difficulty in defining $M_{rn}^{-1/2}$ because $M_{rn} > 0$. To recapitulate, $\mathcal{J}_n = 1$ means completeness, and the aim is to reduce the number of modal coordinates in Q_n as much as possible while maintaining $\mathcal{J}_n$ as close to unity as possible.

The use of the completeness index $\mathcal{J}_n$ is quite straightforward. The basic inertial properties m_n, c_n, and J_n are always known and finding $M_{rn}^{-1/2}$ is straightforward with some minor assistance from a computer. Note that $M_{rn}^{1/2}$ needs to be found once only for each appendage. The coefficients P_n and H_n are each $N_n \times 3$ constant matrices available from modal analysis, and the evaluation of the spectral radius ρ requires nothing more than the solution of a standard eigenvalue problem for a 6×6 real symmetric matrix. So the use of $\mathcal{J}_n$ is clearly quite practical. The implied algorithm is to truncate those modes which least affect $\mathcal{J}_n$.

Spacecraft configurations are sometimes symmetrical in such a manner that individual modes in $\mathcal{E}_n$ contribute to either H_n or P_n but not to both. It is clear directly from Eqs. (7-9) that only modes contributing to H_n are of interest for attitude control. Incompleteness with respect to P_n is of no concern. In such cases, a completeness index based on Eq. (13) alone is indicated

$$\mathcal{J}_n \triangleq \rho[J_n^{-1/2} H_n^T H_n J_n^{-1/2}] \tag{17}$$

where $\mathcal{J}_n$ is the largest of the three real non-negative eigenvalues of $J_n^{-1/2} H_n^T H_n J_n^{-1/2}$. Another equivalent form for Eq. (17) is

$$\mathcal{J}_n = \max_v \{ v^T H_n^T H_n v \} \tag{18}$$

for all 3×1 vectors v such that $v^T J_n v = 1$. For attitude control about a single axis that is dynamically uncoupled from the other two, Eq. (17) simplifies still further to

$$\mathcal{J}_n = J_n^{-1} \sum_{j=1}^{N_n} H_{jn}^2 \tag{19}$$

where J_n and H_{jn} are the parameters associated with the axis in question. On the other hand, if one is interested in the coupling between structural deformations and translational motion (still for symmetrical spacecraft) one would select

$$\mathcal{J}_n = m^{-1} \rho[P_n^T P_n] \tag{20}$$

as the completeness index in place of Eq. (17).

Appendage Modal Frequency Criterion

It is common practice to delete coordinates associated with high-frequency appendage modes. There are several justifications for this. First, the external excitations in space tend to have a low bandwidth. Usually the excitations of highest frequency are caused by controller actions or other internally generated disturbances (e.g., moving parts). It is therefore reasonable to delete modes whose frequencies are significantly higher than the bandwidth of the control system on the basis that, even if present, there is no sustained excitation of them. The second justification is that the modeling of the higher-frequency modes tends to be less reliable than for modes at lower frequencies and, hence, interactions with the former can at best be viewed as slight disturbances on the right side of the motion equations.

One naturally wonders to what extent those two truncation criteria—completeness index and frequency discrimination—are compatible. While no rigorous theory is available for making such a comparison, it can be stated on the basis of numerical experience that the two trends are similar but not always identical. In other words, the modes of lowest frequency *tend* to contribute most to the completeness index $\mathcal{J}_N$. This tendency is clearly true in the limit as very-high-frequency modes are considered. Let H_{jn} and P_{jn} be the modal coefficients for mode j in $\mathcal{E}_n$. In the limit as $j \to \infty$,

$$\Omega_{jn} \to \infty \tag{21}$$

$$\| P_{jn} \| \to 0 \qquad \| H_{jn} \| \to 0 \tag{22}$$

The observations Eqs. (22) follow from Eqs. (11) and (13), which can be written

$$\sum_{j=1}^{\infty} P_{jn} P_{jn}^T = m_n 1 \tag{23}$$

$$\sum_{j=1}^{\infty} H_{jn} H_{jn}^T = J_n \tag{24}$$

When an infinite series converges, it is necessary that the jth term $\rightarrow 0$ as $j \rightarrow \infty$.

Vehicle Modal Coordinate Truncation

Thus far, the system equations for $\mathcal{V}$ have been written in two forms. Equations (1-3) are obtained by structural analysis; Eqs. (7-9) are completely equivalent to Eqs. (1-3), prior to appendage modal truncation, but are written in terms of appendage modal coordinates. Attention is turned now to a third form for the motion equations.

Either Eqs. (1-3) or Eqs. (7-9) can be written

$$\begin{aligned} M_{rr}\ddot{q}_r + M_{re}\ddot{q}_e &= B_{rr}u_r + B_{re}u_e \\ M_{re}^T\ddot{q}_r + M_{ee}\ddot{q}_e + D_e\dot{q}_e + K_e q_e &= B_{ee}u_e \end{aligned} \tag{25}$$

For Eqs. (1-3), the needed definitions on the left side of Eq. (25) are

$$M_{rr} \triangleq \begin{bmatrix} m1 & 0 \\ 0 & I \end{bmatrix} \qquad M_{re} \triangleq \begin{bmatrix} P_{1\delta}^T & \cdots & P_{N\delta}^T \\ H_{1\delta}^T & \cdots & H_{N\delta}^T \end{bmatrix}$$

$$M_{ee} \triangleq \text{diag}\{M_1,\ldots,M_N\} \qquad K_e \triangleq \text{diag}\{K_1,\ldots,K_N\}$$

$$D_e \triangleq \text{diag}\{D_1,\ldots,D_N\} \qquad B_{ee}u_e = [\Upsilon_{1\delta}^T \ldots \Upsilon_{N\delta}^T]^T$$

$$q_r \triangleq [w_c^T \ \theta^T] \quad q_e \triangleq [\delta_1^T \ldots \delta_1^T]^T \quad B_{rr}u_r + B_{re}u_e = [F^T \ G^T]$$

Alternatively, for Eqs. (7-9), we must identify instead

$$M_{re} = \begin{bmatrix} P_1^T \ldots P_N^T \\ H_1^T \ldots H_N^T \end{bmatrix} \qquad M_{ee} = 1 \qquad K_e \triangleq \text{diag}\{\Omega_1^2,\ldots,\Omega_N^2\}$$

$$D_e \triangleq \text{diag}\{\hat{D}_1,\ldots,\hat{D}_N\} \qquad B_{ee}u_e \triangleq [\Upsilon_1^T \ldots \Upsilon_N^T]^T$$

$$q_e \triangleq [Q_1^T \ldots Q_N^T]^T$$

Control actions on $\mathcal{R}$ and $\Sigma\mathcal{E}_n$ have been denoted u_r and u_e. The objective now is to find the modes and frequencies of vibration for $\mathcal{V}$. To this end, note that Eq. (25) can be written

$$M\ddot{q} + D\dot{q} + Kq = Bu \tag{26}$$

where

$$M \triangleq \begin{bmatrix} M_{rr} & M_{re} \\ M_{re}^T & M_{ee} \end{bmatrix} \qquad D \triangleq \begin{bmatrix} 0 & 0 \\ 0 & D_e \end{bmatrix} \qquad B \triangleq \begin{bmatrix} B_{rr} & B_{re} \\ 0 & B_{ee} \end{bmatrix}$$

$$K \triangleq \begin{bmatrix} 0 & 0 \\ 0 & K_e \end{bmatrix} \qquad q \triangleq \begin{bmatrix} q_r \\ q_e \end{bmatrix} \qquad u \triangleq \begin{bmatrix} u_r \\ u_e \end{bmatrix}$$

The system matrices in Eq. (26) have the properties

$$M^T = M > 0 \qquad D^T = D \geq 0 \qquad K^T = K \geq 0 \tag{27}$$

which are transferred to them by their constituent matrix partitions. A transformation T, whose properties are

$$T^T M T = 1 \qquad T^T K T = \hat{K} \tag{28}$$

is now employed to transform Eq. (26) to the form

$$\ddot{\eta} + \hat{D}\dot{\eta} + \hat{K}\eta = T^T Bu \tag{29}$$

where $q = T\eta$ and $\hat{K}$ has the form

$$\hat{K} = \begin{bmatrix} 0 & 0 \\ 0 & \omega^2 \end{bmatrix} \tag{30}$$

Here,

$$\omega \triangleq \text{diag}\{\omega_1,\ldots,\omega_{N_v}\} \tag{31}$$

and $\omega_\alpha > 0$, $\alpha = 1,\ldots,N_v$ are the natural frequencies of vibration for $\mathcal{V}$. The first six diagonal elements in $\hat{K}$ are zeros, corresponding to the rigid modes.

The question at issue is how many coordinates one can afford to include at each successive stage. It has already been pointed out that the number of coordinates in Eqs. (1-3) is given by Eq. (4) where N_n is the number of deformational coordinates assigned to $\mathcal{E}_n$ (that is, the number of entries in δ_n). If Eqs. (1-3) are transformed diretly to Eq. (29), without the intermediate stage of Eqs. (7-9), the number of vehicle-mode coordinates is

$$N_v = \sum_{n=1}^{N} N_n \tag{32}$$

plus, of course, the six rigid-body modes.

The purpose of the intervening transformation to appendage-mode coordinates is to provide an *early* opportunity for model reduction following the guidelines laid down in the last section. In this process, the original N_n coordinates originally associated with $\mathcal{E}_n$ are reduced in number to $\bar{N}_n < N_n$. After transforming to Eq. (29) the number of nonrigid coordinates is, instead of Eq. (32),

$$N_v = \sum_{n=1}^{N} \bar{N}_n \tag{33}$$

Suppose, for example, that $\mathcal{V}$ has eight appendages ($N = 8$), each modeled with 100 degrees of freedom ($N_n = 100$). It may not be possible or desirable to keep all 806 degrees of freedom when transforming to vehicle-mode coordinates. By first reducing the model of each appendage down to $\bar{N}_n = 20$, say, the final transformation is carried out using only 166 configuration variables. This first phase of reduction is the purpose of the intermediate model Eqs. (7-9). If no truncation of appendage-mode coordinates is intended, this intermediate model loses its raison d'être and should not be made. One should go directly from Eqs. (1-3) to Eq. (29).

For the remainder of this section, we consider methods of reducing the model Eq. (29), expressed in vehicle-mode coordinates.

Vehicle Modal Completeness Index

To discuss a reduction in the order of Eq. (29), we first express this model in partitioned form. We set

$$\eta = \begin{bmatrix} \eta_r \\ \eta_e \end{bmatrix} \qquad \hat{D} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{D}_e \end{bmatrix} \qquad T = [T_r \quad T_e]$$

leading to a distinction between the rigid and flexible modes

$$\ddot{\eta}_r = T_r^T(B_{rr}u_r + B_{re}u_e) \tag{34}$$

$$\ddot{\eta}_e + \hat{D}_e\dot{\eta}_e + \omega^2\eta_e = T_e^T(B_{ee}u_e) \tag{35}$$

Moreover, to reduce the level of detail, but at no essential loss in technique, it is assumed from here onward that η_r consists of only the attitude variables $\boldsymbol{\theta}$. It has been shown[10] that T has the form

$$T = \begin{bmatrix} I^{-1/2} & \theta_1 \ldots \theta_{N_v} \\ 0 & t_1 \ldots t_{N_v} \end{bmatrix} \tag{36}$$

The interpretation of θ_α is that its three elements are the rotation angles of $\mathcal{R}$ in vehicle mode α; t_α are the constants found when expressing the αth vehicle mode shape as a linear combination of appendage mode shapes. With these preliminaries explained, the system of Eqs. (34) and (35) becomes

$$\ddot{\eta}_r = I^{-1/2}(B_{rr}u_r + B_{re}u_e) \tag{37}$$

$$\ddot{\eta}_\alpha + \sum_{\beta=1}^{N_v} \hat{d}_{\alpha\beta}\dot{\eta}_\beta + \omega_\alpha^2\eta_\alpha = \theta_\alpha^T B_{rr}u_r + (\theta_\alpha^T B_{re} + \theta_\alpha^T B_{ee})u_e \tag{38}$$

where $\eta_e = [\eta_1 \ldots \eta_{N_v}]^T$ and $\{\hat{d}_{\alpha\beta}\}$ are the elements of the damping matrix $\hat{D}_e$. One may find the attitude of $\mathcal{R}$ from $q = T\eta$

$$\theta = I^{-1/2}\eta_r + \sum_{\alpha=1}^{N_v} \theta_\alpha \eta_\alpha \tag{39}$$

To form an inertial completeness index for the model Eqs. (37) and (38), we note from Ref. 11 that

$$\sum_{\alpha=1}^{\infty} h_\alpha h_\alpha^T = I I_r^{-1} I - I \tag{40}$$

where $h_\alpha \dot{\eta}_\alpha$ is the angular momentum associated with mode α and h_α is related to θ_α by

$$h_\alpha + I\theta_\alpha = 0 \tag{41}$$

The inertia matrix for $\mathcal{R}$ is I_r. The ∞ upper limit in Eq. (40) is meant to imply that the identity is true only for a complete set of modes. Inserting Eq. (41) into Eq. (40), we find an alternative form for the identity

$$\sum_{\alpha=1}^{\infty} \theta_\alpha \theta_\alpha^T = I_r^{-1} - I^{-1} = I_r^{-1} I_e I^{-1} \tag{42}$$

where $I_e \triangleq I - I_r$ is the contribution to I made by the appendages. It is also apparent from Eq. (42) that the series fails to converge if $\mathcal{R}$ shrinks to a point, $I_r \to 0$. This does not imply that $\theta_\alpha \not\to 0$ as $\alpha \to \infty$, but it does imply that more modes are required for a given completeness index if $\mathcal{R}$ is small.

An inertial completeness index can be formulated based on either Eq. (40) or (42); we choose Eq. (42). For an incomplete set of modes, the matrix $I_r^{-1} - I^{-1} - \Sigma\theta_\alpha\theta_\alpha^T$ is positive definite—the more incomplete, the more positive definite. The completeness index $\mathcal{I}$ is defined as follows:

$$\mathcal{I} \triangleq \rho\left[I_r^{1/2}\left(\sum_{\alpha=1}^{N_v} \theta_\alpha\theta_\alpha^T + I^{-1}\right)I_r^{1/2}\right] \tag{43}$$

The manner of using $\mathcal{I}$ is the same as for the appendage modal completeness index $\mathcal{I}_n$, defined in Eq. (16) and discussed thereafter. According to this index, the vehicle modes to be retained are those that contribute most to $\mathcal{I}$. As $\mathcal{I}$ nears unity, it is known that *any* remaining mode must have a very small θ_α. It follows from Eq. (38) that η_α cannot be much excited by θ_α terms and, hence, that this mode is a candidate for deletion based upon *inertial* considerations. It can happen, however, that the frequency of a mode with small θ_α can be near a critical frequency. The following sections include frequency information in truncation decisions.

Truncation Rationale Based on Transfer Functions

Another approach is to examine the transfer functions for Eqs. (37-39). To simplify the argument, it is assumed that the modal damping matrix $\hat{D}_e$ is diagonal

$$\hat{d}_{\alpha\beta} = 2\zeta_\alpha\omega_\alpha\delta_{\alpha\beta} \tag{44}$$

The transfer functions between $\theta(s)$ and $u_r(s)$ or $u_e(s)$ are readily obtained. From Eq. (39),

$$\theta(s) = I^{-1/2}\eta_r(s) + \sum_{\alpha=1}^{N_v} \theta_\alpha\eta_\alpha(s) \tag{45}$$

One next substitutes for η_r and η_α from the Laplace-transformed versions of Eqs. (37) and (38) to obtain

$$s^2\theta(s) = V(s)B_{rr}u_r + [V(s)B_{re} + V_{ee}(s)B_{ee}]u_e \tag{46}$$

where

$$V(s) \triangleq I^{-1} + \sum_{\alpha=1}^{N_v} \frac{s^2\theta_\alpha\theta_\alpha^T}{s^2 + 2\zeta_\alpha\omega_\alpha s + \omega_\alpha^2} \tag{47}$$

$$V_{ee}(s) \triangleq \sum_{\alpha=1}^{N_v} \frac{s^2\theta_\alpha t_\alpha^T}{s^2 + 2\zeta_\alpha\omega_\alpha s + \omega_\alpha^2} \tag{48}$$

For a rigid spacecraft, $V(s) = I^{-1}$ and $V_{ee} = 0$. Concentrating on Eq. (47), we once again see the combination $\theta_\alpha\theta_\alpha^T$ [cf. the completeness index defined in Eq. (43)].

One is interested in the size of the contributions to $V(s)$ from mode α. This contribution can be bounded for lightly damped structures ($\zeta_\alpha \to 0$)

$$\left|\frac{s^2}{s^2 + 2\zeta_\alpha\omega_\alpha s + \omega_\alpha^2}\right| \leq \frac{1}{2\zeta_\alpha} \tag{49}$$

the maximum being attained when $s = j\omega_\alpha$. This maximum is effectively not reached, however, if $\omega_\alpha \gg \omega_{bw}$, where ω_{bw} is a measure of the closed-loop bandwidth (i.e., in the closed loop, the frequency content above ω_{bw} is greatly reduced). Thus, if $\omega_\alpha \gg \omega_{bw}$, and $\zeta_\alpha \ll 1$, the bound in Eq. (49) can be replaced by $(\omega_{bw}/\omega_\alpha)^2$. To ascertain whether mode α makes a sizable contribution to the transfer function $V(s)$, the corresponding term in Eq. (47) must be compared to I^{-1}. Premultiplying and postmultiplying by $I^{1/2}$, we can agree to delete a mode whenever

$$\left|\frac{s^2}{s^2 + 2\zeta_\alpha\omega_\alpha s + \omega_\alpha^2}\right| \rho[I^{1/2}\theta_\alpha\theta_\alpha^T I^{1/2}] \ll 1 \tag{50a}$$

where ρ is once again the spectal norm. Noting that the matrix in Eq. (50a) has eigenvalues $\{0, 0, \theta_\alpha^T I\theta_\alpha\}$, criterion Eq. (50a) has the more explicit form

$$\theta_\alpha^T I\theta_\alpha \ll \begin{cases} (\omega_\alpha/\omega_{bw})^2 & \text{if} \quad \omega_\alpha \gg \omega_{bw} \\ 2\zeta_\alpha & \text{if} \quad \omega_\alpha \text{ is not } \gg \omega_{bw} \end{cases} \tag{50b}$$

This criterion is an attractive one because it includes a consideration of modal frequencies relative to controller bandwidth, modal eigenvectors appropriately through $\theta_\alpha^T I\theta_\alpha$, and modal damping factors ζ_α. Stated in words, criterion Eq. (50) avers that a mode can be truncated for attitude control problems only if 1) the frequency is sufficiently beyond the controller bandwidth, or 2) its interaction with $\mathcal{R}$ as measured by $\theta_\alpha^T I\theta_\alpha$ is sufficiently small, or 3) its damping is sufficiently high. Criterion Eq. (50) can also be thought of as an important generalization of the oft-used truncation criteria based on frequency alone: $\omega_{bw}/\omega_\alpha \ll 1$. A corollary to Eq. (50) is that the spacecraft can be considered rigid for control-design purposes if

$$\max_\alpha\left[\left(\frac{\omega_{bw}}{\omega_\alpha}\right)^2 \frac{\theta_\alpha^T I\theta_\alpha}{\zeta_\alpha}\right] \ll 1 \tag{51}$$

Considerations Based on Frequencies ω_α

In a previous section several justifications were given for deleting appendage modes above a certain frequency. The same justifications apply for dropping high-frequency vehicle modes. Indeed, they apply more so. It is the vehicle modes that are excited directly by forces and torques on the spacecraft.

The omission of high-frequency modes is corroborated by the criteria derived earlier in this section. These higher modes tend to have small θ_α and thus contribute little to the completeness index $\mathcal{I}$ that is, by this time, nearly unity. Criterion Eq. (49) provides even stronger evidence for the soundness of this procedure; the left side of Eq. (49) becomes smaller both as ω_α increases and as $\theta_\alpha^T I \theta_\alpha$ decreases.

An interesting variation on this theme has been studied.[11] Realizing that the vehicle-mode frequencies ω_α computed from the model Eqs. (7-9) depend on the number of appendage modes retained, $(\bar{N}_1,\ldots,\bar{N}_n)$, Likins et al. obtained analytical bounds for the relative errors in ω_α caused by the appendage mode truncation $N_n \to \bar{N}_n$. Thus these bounds provide criteria for appendage mode truncation. It is important to note that these error estimates for ω_α were obtained in literal form, and that the eigenvalue problem for $\mathcal{V}$ does not have to be solved to use these estimates.

Criteria Based on Controllability and Observability

Vehicle-mode truncation can also be approached from the quite different perspective of controllability and observability. With reference to the model Eqs. (37) and (38), it is shown in Ref. 10 that the open-loop controllability of mode α is proportional to the modal controllability

$$\mathcal{C}_\alpha \triangleq \| \theta_\alpha^T B_{rr}, \theta_\alpha^T B_{re} + t_\alpha^T B_{ee} \| \tag{52}$$

assuming the ω_α are distinct. It is important to note that $\mathcal{C}_\alpha$ depends not only on modal data $(\theta_\alpha, t_\alpha)$ but also on the locations of the actuators (in B_{re}, B_{ee}). The most important use of Eq. (52) is therefore to position actuators to maximize the $\mathcal{C}_\alpha$ of those modes one wishes to control and perhaps to reduce the controllability of other modes. Once the actuators are fixed in location, however, it is reasonable at least for control system design, to delete modes which have a small $\mathcal{C}_\alpha$ and that therefore cannot be effectively controlled anyway. For control system evaluation, these modes should be reinserted.

Observability can also be used in an analogous fashion. For the first time in this paper, we need to introduce the inputs (measurements) to the controller; these are denoted by y

$$y = Pq + P'\dot{q} = \hat{P}\eta + \hat{P}'\dot{\eta} \tag{53}$$

where $\hat{P} = PT$, $\hat{P}' = P'T$. The structure of P and P' is as follows:

$$P = \begin{bmatrix} P_{rr} & 0 \\ P_{er} & P_{ee} \end{bmatrix} \qquad P' = \begin{bmatrix} P'_{rr} & 0 \\ P'_{er} & P'_{ee} \end{bmatrix} \tag{54}$$

It is shown in Ref. 10 that the measurement observability (measurability for short) of mode α is proportional to

$$\mathcal{O}_\alpha^2 \triangleq \| \theta_\alpha^T P_{rr}^T, \quad \theta_\alpha^T P_{er} + t_\alpha^T P_{ee}^T \|^2 + \omega_\alpha^2 \| \theta_\alpha^T P_{rr}'^T, \quad \theta_\alpha^T P_{er}'^T + t_\alpha^T P_{ee}'^T \|^2 \tag{55}$$

The chief use of Eq. (55) is to assist in sensor placement. Once the sensors are placed, modes with small $\mathcal{O}_\alpha$ may be deleted for control design on the grounds that the controller is not aware of them anyway.

Further discussion on controllability and observability as truncation criteria for flexible spacecraft can be found in Refs. 12 and 13.

Criteria Based on Modal Cost Analysis

In order to use modal cost analysis[9] as a basis for order reduction, it is necessary to specify a "cost" functional

$$V \triangleq \mathrm{E} \int_0^\infty y^T Q y \, \mathrm{d}t \tag{56}$$

where E means expected value. Here, y is some output to be minimized and need not now denote measurments. Assuming lightly damped structures ($\zeta_\alpha \ll 1$) it has been shown[14,15] that

$$V = \sum_{\alpha=1}^{N_v} V_\alpha \tag{57}$$

where

$$V_\alpha = \left(\frac{p_\alpha^T Q p_\alpha + \omega_\alpha^2 p_\alpha'^T Q p_\alpha'}{4\zeta_\alpha \omega_\alpha^3} \right) \sigma_\alpha^2 \tag{58}$$

$$p_\alpha \triangleq \begin{bmatrix} P_{rr}\theta_\alpha \\ P_{er}\theta_\alpha + P_{ee}t_\alpha \end{bmatrix} \quad p_\alpha' \triangleq \begin{bmatrix} P'_{rr}\theta_\alpha \\ P'_{er}\theta_\alpha + P'_{ee}t_\alpha \end{bmatrix} \tag{59}$$

$$\sigma_\alpha^2 \triangleq \mathrm{E}\{ \dot{\eta}_\alpha^2(0) \} \tag{60}$$

Comparing Eq. (58) to Eq. (55), it can be seen that when $Q = 1$,

$$V_\alpha = \mathcal{O}_\alpha^2 \sigma_\alpha^2 / 4\zeta_\alpha \omega_\alpha^3 \tag{61}$$

where now $\mathcal{O}_\alpha$ is the performance observability of mode α. The modal-cost philosophy[9] is to retain modes with high V_α. Equation (61) shows that we should keep the least damped, lowest frequency, most disturbed, most performance-observable modes.

It is interesting to compare criterion Eq. (61) with criterion Eq. (50). For attitude control, with $y = \Sigma \theta_\alpha \eta_\alpha$ and $p_\alpha \equiv \theta_\alpha$, $p_\alpha' \equiv 0$, Eq. (58) becomes[14]

$$V_\alpha = (\theta_\alpha^T \theta_\alpha / 4\zeta_\alpha \omega_\alpha^3)\sigma_\alpha^2 \tag{62}$$

There is clearly a large measure of agreement between Eqs. (62) and (50). Both suggest that modes with very small θ_α can be dropped. Both require that very lightly damped modes be retained. And, considering that the disturbances contained in σ_α^2 will tend to be very negligible well above the controller bandwidth, they both agree that frequencies well above the controller bandwidth can be dropped.

There is another message in both Eqs. (62) and (50). For assessing the importance of the interaction between the controller and mode α, it is as important to know ζ_α as it is to know ω_α. This provides a strong motivation for spending more effort on determining ζ_α than has heretofore been spent.

For vibration suppression problems, one might seek to minimize the total energy (potential plus kinetic) of the vibrating structure[14]

$$y^T Q y = \sum_{\alpha=1}^{N_v} (\dot{\eta}_\alpha^2 + \omega_\alpha^2 \eta_\alpha^2) \tag{63}$$

whence $p_\alpha^T Q p_\alpha = \omega_\alpha^2$, $p_\alpha'^T Q p_\alpha' = 1$, and Eq. (58) becomes

$$V_\alpha = \sigma_\alpha^2 / 2\zeta_\alpha \omega_\alpha \tag{64}$$

Again, modes with lowest V_α are the first to be discarded.

Finally, for figure control,[14] one may choose

$$y^T Q y = \sum_{\alpha=1}^{N_v} a_\alpha \eta_\alpha^2 \qquad (a_\alpha \geq 0) \tag{65}$$

whence $p_\alpha^T Q p_\alpha = a_\alpha$, $p_\alpha'^T Q p_\alpha' = 0$, where a_α is a geometrical indicator of the importance of mode α to the quantity to be minimized. The modal cost Eq. (58) is now

$$V_\alpha = a_\alpha \sigma_\alpha^2 / 4\zeta_\alpha \omega_\alpha^3 \tag{66}$$

Not surprisingly, modes with high a_α should be kept (other things being equal) so that the control system, when added to the model, can concentrate on these modes.

Conclusions

Many concepts and criteria for model-order reduction have been discussed in this paper. All are based on mode truncation. To sort out these criteria and to identify the situations in which they are most useful, it might be helpful briefly to retrace the path followed in the preceding development. A basic spacecraft model of considerable generality was first presented. With several flexible appendages, some reduction in model order can be achieved if necessary by transforming to appendage modal coordinates and using the completeness index defined in its most general form by Eq. (16). At this stage it is assumed that the ultimate purpose of the model has not been specified precisely, and that nothing is known about the control system. Therefore a completeness index quite close to unity is desirable. Using this criterion will not produce the same model as a simple frequency criterion but for a given number of modes the model produced will be more complete inertially. Although only modal data are used, an algorithm is required to select those modes that contribute most to the completeness index. No truncation should be made at this stage unless the eigenvalue problem for the whole vehicle is unacceptably large.

Once the model is in vehicle modal coordinates, further model reduction is possible. A new completeness index $\mathcal{I}$ was formulated, specialized now to attitude control, and given by Eq. (43). Confidence in the model grows as $\mathcal{I}$ rises. The traditional procedure of dropping modes with high ω_α is also defensible, especially if done after a high $\mathcal{I}$ has been achieved. "High ω_α" means "high compared to the control-system bandwidth."

A new criterion was then advanced, Eq. (50), that takes into account not only eigenvalues (ω_α) and eigenvectors ($\boldsymbol{\theta}_\alpha$) but also the level of dissipation in each. It becomes quite clear that modal damping factors (ζ_α) must be known with considerable precision if modal truncation decisions are to be reliable.

If the location and effectiveness of control actuators are known, the modal controllability $\mathcal{C}_\alpha$ can be used as a criterion. Similarly, if the location and properties of the sensors are known, the measurability of mode α, $\mathcal{O}_\alpha$, can be taken into account. See Eqs. (52) and (55).

Finally, if a cost index V is posited, greater specificity in order-reduction decisions is possible. For lightly damped structures, a portion of the cost V can be assigned to mode α. This modal cost V_α has been expressed in terms of the performance observability $\mathcal{O}_\alpha$, the disturbance level σ_α, the natural frequency ω_α, and the damping ratio ζ_α is again in evidence and, other things being equal, modes with low performance observability can be dropped from the model. Equation (61) is typical of this criteria.

The criteria in this paper are not intended to replace experienced engineering judgment but to provide a quantitative basis for it. It is hoped that future spacecraft experiences will further elucidate the application of these criteria and identify their strengths and limitations.

Still further reductions in model order may be necessary once the feedback control system is in place but these techniques are beyond the scope of this paper.

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